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The viscous nonlinear symmetric baroclinic instability of a zonal shear flow

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The stability of a baroclinic zonal current to symmetric perturbations on a meridionally unbounded f-plane is considered. The lower boundary is at rest but the upper one moves with a constant velocity in keeping with the velocity of the zonal current. Following Stone (1966) a horizontal length scale $O(Ro)$ is taken, where Ro is the Rossby number, with the Richardson number $Ri = O(1)$. Instability sets in when the wavelength is $O(E^{\frac{1}{3}})$, where E is the Ekman number based on the distance between the rigid horizontal boundaries, which corresponds to Stone's inviscid value zero, and to McIntyre's (1970) value infinity on a length scale $O(E^{\frac{1}{2}})$.

A nonlinear analysis about the point of onset of instability yields the result that for the monotonic mode zonal momentum is convected polewards. The possible implications of this result for the dynamics of Jupiter's atmosphere are discussed.

1. Introduction

The baroclinic stability of a zonal flow with vertical shear has attracted much attention since Eady's (1949) classical f-plane analysis. He assumed a strongly stratified atmosphere, i.e. one in which the Richardson number $Ri \gg 1$, and was able to relate the growing instabilities to the development of cyclones in the earth's atmosphere. Stone (1966) extended Eady's model to include moderate values of Ri and found that symmetric instabilities (those independent of longitude) are possible if $Ri < 1$ and, at least in a linear model, have the greatest growth rates when $0{\cdot}25 < Ri < 0{\cdot}95$ (see also Eliassen & Kleinschmidt 1957; Charney 1973 and references). In subsequent papers Stone (1967, 1972) and Gierasch & Stone (1968) have developed the attractive hypothesis that symmetric instabilities of a zonal shear flow may account for some features of Jupiter's atmosphere, chiefly its symmetric cloud bands and its equatorial jet (see Peek 1958). In order to maintain the jet angular momentum must be convected equatorwards and one of Stone's interesting results is that the symmetric instabilities provide a mechanism for this provided that $\frac{1}{4} < Ri < \frac{1}{3}$. On the other hand, Hide (1970) argues from physical considerations that they cannot do so whatever the value of Ri.

There are two features of Stone's analysis which merit further study and which

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we shall discuss in this paper. The first is the fact that for an inviscid fluid the critical wavelength for the onset of instability is zero. This strongly suggests that a new length scale is required in this neighbourhood, and by introducing the viscosity ν and conductivity κ (here assumed equivalent for scaling purposes) we may obtain any length scale we wish by combining appropriate powers of $(\nu/f)^{\frac{1}{2}}$ and H, where $\frac{1}{2}f$ is the angular velocity of the system and H is the depth of the atmosphere. McIntyre (1970), in a related problem, set the boundaries at infinity and used the only possible length scale, $(\nu/f)^{\frac{1}{2}}$. He found that when the basic flow has no horizontal shear instability first sets in at $Ri = Ri_c = (1+\sigma)^2/4\sigma$ (> 1 when $\sigma \neq 1$), where $\sigma = \nu/\kappa$ is the Prandtl number, when the wavelength is infinite. The destabilization of certain classically stable modes by the introduction of viscosity and thermal conductivity is an interesting feature of the solution [similar results have been obtained by Yih (1959, 1961) for viscosity and electrical conductivity; see also Acheson & Hide (1973)], but from our point of view the most significant feature is that yet another length scale is required if we are to discuss the onset of instability.

In §2 of this paper we show that the appropriate length scale for marginal instability is $O((\nu/H^2f)^{\frac{1}{3}})$ and that instability sets in, at a finite wavelength, when the Richardson number is that given by McIntyre (*loc. cit.*) less a term $O((\nu/H^2f)^{\frac{2}{3}})$.

The second feature is that a linear stability analysis was applied by Stone to the basic zonal flow when $\Delta = Ri_c - Ri$ is no longer small. For these values of Ri the flow may be thought of as being set up instantaneously, but this leads to a complicated mathematical problem in which a continuous spectrum of modes are simultaneously unstable. The basic zonal flow is destroyed within a few rotations of the planet. A more informative approach is to suppose that instability is set up as Ri decreases from a supercritical (stable) value through the critical value. The flow after the onset of instability is likely to be nonlinear and the linear theory is likely to be inadequate. The first question to be considered by such a theory is whether the flow evolves from the unstable form of a uniform zonal shear to a new stable form or becomes catastrophically unstable and turbulent.

This question is answered in §3 for $\Delta \ll 1$ (but not infinitesimal) by using a nonlinear analysis similar to that of Stuart (1958, 1960) about the point of onset of monotonic instability (this mode has a higher critical Richardson number than the oscillatory instabilities which are also possible). The result is that the amplitude of the disturbance tends to a constant finite value. We interpret this result as implying an exchange of stabilities at $\Delta = 0$ and that for $\Delta > 0$ there exists a stable solution of the problem in which the meridional velocity v is non-zero and such that $v \to 0$ as $\Delta \to 0$. The form of this solution has been found for $\Delta \ll 1$ and further study is needed to obtain its form at more moderate values of Δ. As Δ increases this stable solution evolves and it is possible that at some stage $\Delta = \Delta_1$, say, stability is lost and a further bifurcation to a more complicated form or even a catastrophic instability in which there is no stable solution for $\Delta > \Delta_1$ occurs. The theory of Taylor-vortex flow (Stuart 1971) seems to imply that such a history is possible. This theory also provides some evidence that the flow is stable to nonlinear as well as linear disturbances when Ri is just supercritical.

Finally, in § 4, the momentum flux is shown to be polewards for all values of the Prandtl number and for Richardson numbers not far removed from the critical value in agreement with Hide's and Stone's results.

2. The linear stability problem

Stone's (1972) model assumes a Boussinesq adiabatic fluid contained between two horizontal planes at $z^* = 0$, H but unbounded horizontally. Rectangular Cartesian co-ordinates (x^*, y^*, z^*) are taken in the zonal, meridional and vertical directions and the motion takes place on an f-plane rotating about a vertical axis with angular velocity $\frac{1}{2}f$. Let the fluid have density ρ^*, pressure p^*, velocity (u^*, v^*, w^*) in the (x^*, y^*, z^*) co-ordinates, temperature θ^*, kinematic viscosity ν and thermometric conductivity κ. Then the conservation equations are

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*} = 0, \tag{2.1}$$

$$\frac{du^*}{dt^*} = fv^*-\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*}+\nu\nabla_*^2 u^*, \tag{2.2}$$

$$\frac{dv^*}{dt^*} = -fu^*-\frac{1}{\rho^*}\frac{\partial p^*}{\partial y^*}+\nu\nabla_*^2 v^*, \tag{2.3}$$

$$\partial p^*/\partial z^* = \alpha\rho^* g\theta^*, \quad d\theta^*/dt^* = \kappa\nabla_*^2\theta^*, \qquad (2.4), (2.5)$$

where g is the acceleration due to gravity, t^* measures time and we have already anticipated that $\partial/\partial z^* \gg \partial/\partial x^*$, $\partial/\partial y^*$ and $w^* \ll u^*, v^*$. These equations differ from Stone's only in the inclusion of the diffusive terms. The boundary conditions are

$$u^* = v^* = w^* = 0 \quad \text{at} \quad z^* = 0, H. \tag{2.6}$$

The basic flow is assumed to consist of a zonal wind of magnitude U with constant vertical shear and a temperature field with constant vertical stratification $\partial\theta_0^*/\partial z^*$ related to U by the thermal-wind equation. We non-dimensionalize with reference to this state by writing

$$(x^*, y^*, z^*) = ((U/f)x, (U/f)y, Hz), \quad (u^*, v^*, w^*) = (Uu, Uv, fHw),$$
$$\theta^* = H(\partial\theta_0^*/\partial z^*)\theta, \quad t^* = f^{-1}t, \quad p^* = \alpha\rho^* gH^2(\partial\theta_0^*/\partial z^*)p.$$

Then (2.1)–(2.5) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{2.7}$$

$$\frac{du}{dt} = v-Ri\frac{\partial p}{\partial x}+E\frac{\partial^2 u}{\partial z^2}, \quad \frac{dv}{dt} = -u-Ri\frac{\partial p}{\partial y}+E\frac{\partial^2 v}{\partial z^2}, \qquad (2.8), (2.9)$$

$$\frac{\partial p}{\partial z} = \theta, \quad \frac{d\theta}{dt} = \frac{E}{\sigma}\frac{\partial^2\theta}{\partial z^2}, \qquad (2.10), (2.11)$$

where $$Ri = \frac{\alpha g H^2\, \partial\theta_0^*/\partial z^*}{U^2} \quad \text{(Richardson number)},$$

$$E = \nu/H^2 f = \nu f/\epsilon^2 U^2 \quad \text{(Ekman number)},$$

$$\sigma = \nu/\kappa \quad \text{(Prandtl number)},$$

$$Ro = U/af \quad \text{(Rossby number)},$$

$$\epsilon = Hf/U = H/aRo \quad \text{(aspect ratio)}.$$

Here a is some horizontal reference length scale. In deriving these equations we have assumed that the ratio of vertical to horizontal length scales is small compared with the Richardson number, i.e., $\epsilon \ll Ri$. We shall further assume that $Ri = O(1)$ and $\epsilon \ll 1$.

Then the basic zonal flow may be taken to be

$$v_0 = w_0 = 0, \quad u_0 = z, \quad \theta_0 = z - y/Ri, \quad p_0 = \tfrac{1}{2}z^2 - yz/Ri. \tag{2.12}$$

In order to avoid the complication of an Ekman layer at the upper boundary we shall suppose that it moves with a dimensionless velocity of unity; (2.12) then satisfies the viscous boundary conditions.

When deviations from this steady state have amplitude Δ with $\Delta \ll 1$ we may expand the total solution in powers of Δ:

$$\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}_1 + \Delta^2\mathbf{u}_2 + \ldots, \text{ etc.} \tag{2.13}$$

The first-order equations are obtained by substituting (2.13) into (2.7)–(2.11) and equating coefficients of Δ. We are interested here in the symmetric instabilities of the zonal flow, which means that the perturbed flow is taken to be independent of x and the linearized problem to have normal-mode solutions of the form

$$u_1 = u_1(z)\exp\{ily + \omega t\} \quad \text{etc.},$$

where

$$ilv_1 + dw_1/dz = 0, \tag{2.14}$$

$$\omega u_1 + w_1 = v_1 + E\, d^2u_1/dz^2, \tag{2.15}$$

$$\omega v_1 + u_1 = -ilRip_1 + E\, d^2v_1/dz^2, \tag{2.16}$$

$$\partial p_1/\partial z = \theta_1, \tag{2.17}$$

$$\omega\theta_1 - v_1/Ri + w_1 = (E/\sigma)\, d^2\theta_1/dz^2. \tag{2.18}$$

These equations may be reduced to

$$D_\sigma D^2 \frac{d^2w_1}{dz^2} + D_\sigma \frac{d^2w_1}{dz^2} + il(D_\sigma + D)\frac{dw_1}{dz} - l^2 Ri D w_1 = 0, \tag{2.19}$$

where $D \equiv \omega - E d^2/dz^2$ and $D_\sigma \equiv \omega - E/\sigma\, d^2/dz^2$. Equation (2.19), together with the boundary conditions

$$w_1 = u_1 = v_1 = \theta_1 = 0 \quad \text{at} \quad z = 0, 1, \tag{2.20}$$

defines the eigenvalue problem for the complex frequency ω. The basic flow is then unstable to axisymmetric disturbances if $\mathrm{Re}\,\omega > 0$, and we are particularly interested here in determining the neutral-stability curve $\mathrm{Re}\,\omega = 0$.

The inviscid problem defined by setting $E = 0$ in (2.19) is

$$\omega[(1+\omega^2)\, d^2w_1/dz^2 + 2il\, dw_1/dz - l^2 Ri w_1] = 0, \quad w_1 = 0 \quad \text{at} \quad z = 0, 1,$$

and apart from the trivial solution $\omega = 0$ corresponds to Stone's (1972) equation (2.25). It is then easily shown that

$$\omega^2 = -\left\{1+\frac{Ril^2}{2m^2\pi^2}\left[1-\left(1+\frac{4m^2\pi^2}{Ri^2l^2}\right)^{\frac{1}{2}}\right]\right\}, \quad m = 1,2,3,\ldots, \tag{2.21}$$

i.e. that $\omega^2 = 0$ when $l = m\pi/(1-Ri)^{\frac{1}{2}}$. Hence the flow is stable for $Ri > 1$ and instability first occurs at $Ri = 1$ as $l \to \infty$. The most unstable mode (with maximum $\mathrm{Re}\,\omega$) also occurs as $l \to \infty$ and this suggests that a closer examination of these higher wavenumbers is required. No new scaling is possible in the inviscid model but by turning our attention to the viscous equations and supposing that E is small we may look for solutions of the form

$$w_1 = \exp\{\omega t + iE^{-\frac{1}{2}}(ly+\lambda z)\}.$$

Then (2.19) becomes

$$\lambda^2(\omega+\lambda^2/\sigma)(\omega+\lambda^2)^2+\lambda^2(\omega+\lambda^2/\sigma)+\lambda l[2\omega+(1+\sigma^{-1})\lambda^2]+l^2(\omega+\lambda^2)Ri = 0, \tag{2.22}$$

and this corresponds to McIntyre's (1970) equation (2.9), differing only in that we have taken $\epsilon \ll 1$, so that his k is equivalent to our λ. Certain properties of this equation have been obtained and discussed by McIntyre (*loc. cit.*) and we shall briefly summarize some of them here. There is a real root of (2.22) which, in the inviscid limit $\lambda, l \to 0$, corresponds to the trivial solution $\omega \equiv 0$. This monotonic instability first sets in as λ, $l \to 0$, when

$$Ri = -\sigma^{-1}(\lambda/l)\,[\lambda/l+(1+\sigma)],$$

and Ri is a maximum when $\lambda/l = -\frac{1}{2}(1+\sigma)$, the maximum value being $R_{mc} = (1+\sigma)^2/4\sigma$. McIntyre notes that $R_{mc} > 1$ when $\sigma \neq 1$, which means that although the flow is classically stable it is unstable in the presence of diffusive mechanisms of different magnitudes. The other two roots of (2.22) are complex conjugates and correspond to oscillatory instabilities which are the viscous counterparts of that given in (2.21). Again instability first sets in as $\lambda, l \to 0$, when

$$Ri = \frac{-2\sigma\lambda}{1+\sigma}\frac{\lambda}{l}\left(\frac{\lambda}{l}+\frac{1+3\sigma}{2\sigma}\right),$$

and Ri is a maximum when $\lambda/l = -(1+3\sigma)/4\sigma$, the maximum value being $R_{oc} = (1+3\sigma)^2/8\sigma(1+\sigma)$. This is also greater than 1, so that diffusion is destabilizing here also.

This solution does not satisfy the boundary conditions and indeed for time scales $O(1)$ a localized disturbance of this form will not be affected by the boundaries. However, if we are to extend this linear analysis to nonlinear interactions which occur on a much larger time scale, we may anticipate that the boundaries will play an important part. A nonlinear analysis centred on R_{mc} or R_{oc} will involve the critical wavenumbers, which are both zero, and this is as poor for our purposes as Stone's value of infinity. Both Stone's and McIntyre's scales are inappropriate in this critical neighbourhood, though it should be emphasized that their scales are appropriate for their particular problems and, indeed, they had no other choice of scale. In order to determine a new scale we write

$$w_1 = (\beta_+ e^{i\lambda_+ z}+\beta_- e^{i\lambda_- z})\exp(ily+\omega t),$$

where $\beta_\pm$ are constants. Then the boundary conditions $w_1 = 0$ at $z = 0, 1$ are satisfied if $\beta_- = -\beta_+$ and

$$\lambda_+ - \lambda_- = 2m\pi, \tag{2.23}$$

where m is an integer. Substituting into (2.19) we get

$$\omega^3 + a\omega^2 + b\omega + c = 0, \tag{2.24}$$

where

$$\begin{aligned} a &= E\lambda^2(2+\sigma^{-1}), \\ b &= (1+2l/\lambda + l^2/\lambda^2 Ri) + E^2\lambda^4(1+2\sigma^{-1}), \\ c &= [E(\lambda^2 + l\lambda(1+\sigma) + \sigma l^2 Ri) + E^3\lambda^6]/\sigma \end{aligned}$$

and λ represents λ_+ or λ_-. We now suppose that

$$\left.\begin{aligned} \lambda_\pm &= E^{-\alpha}(\lambda_{0\pm} + E^\alpha \lambda_{1\pm} + \ldots), \\ l &= E^{-\alpha}(l_0 + E^\alpha l_1 + \ldots), \\ Ri &= R_0 + E^\alpha R_1 + E^{2\alpha} R_2 + \ldots, \end{aligned}\right\} \tag{2.25}$$

where α is yet to be determined.

For the monotonic mode the neutral-stability curve is given by $\omega = 0$, i.e. $c = 0$. Substituting (2.25) into (2.24) and assuming that $\alpha < \frac{1}{2}$ we have

$$\lambda_0^2 + (1+\sigma)l_0\lambda_0 + \sigma R_0 l_0^2 = 0,$$

and the boundary condition (2.23) means that this equation must have equal roots, i.e.

$$\left.\begin{aligned} R_0 &= (1+\sigma)^2/4\sigma = R_{mc} \\ \text{and} \qquad \lambda_{0\pm} &= -\tfrac{1}{2}(1+\sigma)l_0. \end{aligned}\right\} \tag{2.26}$$

Continuing the expansion we find that if $\alpha > \frac{2}{7}$

$$\lambda_1[2\lambda_0 + (1+\sigma)l_0] + l_1[(1+\sigma)\lambda_0 + 2\sigma l_0 R_0] + \sigma l_0^2 R_1 = 0,$$

i.e.

$$R_1 = 0.$$

Assuming that $\alpha \leqslant \frac{2}{7}$ leads us back essentially to Stone's problem, i.e. the scaling is too large. Now equating terms in $E^{3-6\alpha}$ with those in E, i.e. taking $\alpha = \frac{1}{3}$, we obtain

$$\lambda_1^2 + (1+\sigma)l_1\lambda_1 + \sigma R_0 l_1^2 + \sigma R_2 l_0^2 + \lambda_0^6 = 0.$$

Applying the conditions (2.23) we have

$$l_1^2[(1+\sigma)^2 - 4\sigma R_0] - 4\sigma R_2 l_0^2 - 4\lambda_0^6 = 4m^2\pi^2, \tag{2.27}$$

i.e.

$$R_2 = -\frac{(1+\sigma)^2}{4\sigma}\left(\frac{\lambda_0^6 + m^2\pi^2}{\lambda_0^2}\right),$$

$$\lambda_{1\pm} = \tfrac{1}{2}[-(1+\sigma)l_1 \pm 2m\pi].$$

It is easily shown that R_2 has a maximum as a function of λ_0 when λ_0 is the positive real root of $\lambda_0^6 = \frac{1}{2}m^2\pi^2$, the maximum being

$$R_{2mc} = -\frac{3}{\sigma}\left(\frac{1+\sigma}{2}\right)^2\left(\frac{m^2\pi^2}{2}\right)^{\frac{2}{3}}. \tag{2.28}$$

For the oscillatory mode $\omega = i\omega_i$ on the neutral-stability curve with ω_i real. Substituting into (2.24) and equating real and imaginary parts to zero we obtain

$$\omega_i^2 = b = c/a,$$

which means that

$$(1+2\sigma)\,[\lambda^2+2l\lambda+l^2Ri+E^2\lambda^6(1+2\sigma^{-1})] = \lambda^2+l\lambda(1+\sigma)+\sigma l^2Ri+E^2\lambda^6.$$

Again using the expansion (2.25) we obtain

$$\lambda_0^2+(1+3\sigma)/2\sigma\lambda_0 l_0+(1+\sigma)/2\sigma R_0 = 0$$

and this has equal roots if

$$\left.\begin{aligned} R_0 &= (1+3\sigma)^2/8\sigma(1+\sigma) = R_{oc}, \\ \lambda_{0\pm} &= -l_0(1+3\sigma)/4\sigma. \end{aligned}\right\} \tag{2.29}$$

Again $R_1 = 0$ provided that $\alpha > \frac{2}{7}$ and, taking $\alpha = \frac{1}{3}$, terms in E give

$$\lambda_1^2+\frac{1+3\sigma}{2\sigma}l_1\lambda_1+\frac{1+\sigma}{2\sigma}R_0 l_1^2+\frac{(1+\sigma)^2}{\sigma^2}\lambda_0^6+\frac{1+\sigma}{2\sigma}R_2 l_0^2 = 0.$$

Satisfying the boundary conditions gives

$$R_2 = -\frac{(1+3\sigma)^2}{8\sigma(1+\sigma)}\left(\frac{m^2\pi^2+(1+\sigma^{-1})^2\lambda_0^6}{\lambda_0^2}\right). \tag{2.30}$$

Again R_2 clearly has a maximum

$$R_{2oc} = -\,3de(m^2\pi^2/2d)^{\frac{2}{3}}$$

when λ_0 is the positive real root of

$$\lambda_0^6 = m^2\pi^2/2d, \tag{2.31}$$

where $$d = (1+\sigma^{-1})^2, \quad e = (1+3\sigma)^2/8\sigma(1+\sigma).$$

In the inviscid model $\omega_i = 0$ on the neutral-stability curve and it is of interest to investigate its value in the critical region. We have

$$\omega_i^2 = b = (3+\sigma)\,(1-\sigma)/(1+3\sigma)\,(1+\sigma)+O(E^{\frac{1}{3}}),$$

so that the effect of viscosity is not only to destabilize the flow but also to cause this mode to oscillate around the point of stability. Incidentally, when $\sigma = 1$, $R_0 = 1$ and $\omega_i^2 = O(E^{\frac{1}{3}})$ and this closely resembles the inviscid solution when we let $E \to 0$.

It is perhaps worthwhile comparing our results with those for rotating Bénard convection (Chandrasekhar 1961). In that problem an inviscid fluid is stable to monotonic disturbances whatever the value of the adverse temperature gradient as a direct result of the Taylor–Proudman theorem. The presence of viscosity is essential for that constraint to be broken and instability to develop. In our problem the vorticity of the basic shear flow inclines the (absolute) vortex lines at an angle to the axis of rotation but the above result still applies and again viscosity is seen to be necessary for the onset of instability. Also, our governing equation (2.19) is similar to that for rotating Bénard convection (Chandrasekhar 1961, p. 104) and it is consequently no surprise that our critical wavelength is $O(E^{\frac{1}{3}})$.

3. The nonlinear disturbance

We consider here a perturbation of the basic zonal flow near the neutral-stability curve when the increment of the Richardson number above its maximum Ri_c on this curve is small. From the linear analysis [equation (2.19)] it may be seen that when ω is small, $O(\lambda^{-2})$, a linear relation exists between ω and Ri, so that when $\Delta = Ri_c - Ri$ is small the growth rate of the disturbance will be $O(\Delta)$. (In the inviscid theory the growth rate is $O(\Delta^{\frac{1}{2}})$ as in the model examined by Pedlosky (1970)). This suggests that we introduce a slow time variable $T = |\Delta| t$. An appropriate expansion is then

$$w = w_0(y,z) + |\Delta|^{\frac{1}{2}} w_1(y,z,t,T) + |\Delta| w_2(y,z,t,T) + \ldots, \quad \text{etc.} \tag{3.1}$$

The governing equations (2.7)–(2.11) then become

$$\partial v/\partial y + \partial w/\partial z = 0, \tag{3.2}$$

$$|\Delta| \frac{\partial u}{\partial T} + \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = v + E\frac{\partial^2 u}{\partial z^2}, \tag{3.3}$$

$$|\Delta| \frac{\partial v}{\partial T} + \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -u - Ri_c\frac{\partial p}{\partial y} + E\frac{\partial^2 v}{\partial z^2} + \Delta\frac{\partial p}{\partial y}, \tag{3.4}$$

$$\partial p/\partial z = \theta, \tag{3.5}$$

$$|\Delta| \frac{\partial \theta}{\partial T} + \frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = \frac{E}{\sigma}\frac{\partial^2 \theta}{\partial z^2}. \tag{3.6}$$

The $O(1)$ solution is the basic flow given in (2.12), in which $Ri = Ri_c - \Delta$, i.e.,

$$\left.\begin{aligned} u_0 &= z, \quad v_0 = w_0 = 0, \\ \theta_0 &= z - y/Ri_c - \Delta y/Ri_c^2 + O(\Delta^2), \\ p_0 &= \tfrac{1}{2}z^2 - yz/Ri_c - \Delta yz/Ri_c^2 + O(\Delta^2), \end{aligned}\right\} \tag{3.7}$$

where the terms in Δ, although not strictly $O(1)$, are retained for convenience.

The $O(|\Delta|^{\frac{1}{2}})$ problem obtained after substituting (3.1) into (3.2)–(3.6) is equivalent to the linear problem already discussed. We shall discuss only the monotonic mode here as it is this that becomes unstable first for all values of $\sigma \neq 1$ as Ri decreases. When $\sigma = 1$ both modes become unstable at $Ri = 1$; this complicated special case will be excluded here. The linear solution is

$$w_1 = E^{\frac{1}{3}} A(T)\, \mathscr{E} \sin m\pi z, \tag{3.8a}$$

$$v_1 = -E^{\frac{1}{3}} A(T)\, (\lambda/l) \mathscr{E} \sin m\pi z + E^{\frac{2}{3}} A(T)\, (im\pi/l) \mathscr{E} \cos m\pi z, \tag{3.8b}$$

$$\begin{aligned} u_1 = &-A(T)\left(\frac{\lambda + l}{\lambda^2 l} + O(E^{\frac{2}{3}})\right) \mathscr{E} \sin m\pi z \\ &- E^{\frac{1}{3}} A(T)\left(\frac{im\pi(2l+\lambda)}{l\lambda^3} + O(E^{\frac{2}{3}})\right) \mathscr{E} \cos m\pi z, \end{aligned} \tag{3.8c}$$

$$\theta_1 = -A(T)\left(\frac{\sigma(\lambda + Ri_c l)}{Ri_c \lambda^2 l} + O(E^{\frac{2}{3}})\right)\mathscr{E}\sin m\pi z$$
$$-E^{\frac{1}{3}}A(T)\left(\frac{i\sigma m\pi(\lambda + 2lR_c)}{Ri_c l\lambda^3} + O(E^{\frac{2}{3}})\right)\mathscr{E}\cos m\pi z, \qquad (3.8\,d)$$

where $\mathscr{E} \equiv \exp\{iE^{-\frac{1}{3}}(ly + \lambda z)\}$ and l and λ take their critical values.

Although $w_1 = 0$ at $z = 0, 1$ it is apparent that the terms in $\cos m\pi z$ in u_1 and v_1 prevent us from satisfying the no-slip condition at the upper and lower boundaries. An investigation of the Ekman layers set up at these boundaries indicates that the correction to the single mode under examination is $O(E)$ in the w component, and we may neglect this in the subsequent analysis. However, if we were to consider the nonlinear interactions of different modes we would have to take into account an $O(E^{\frac{1}{2}})$ contribution from the Ekman layers.

The $O(|\Delta|)$ problem is

$$\partial v_2/\partial y + \partial w_2/\partial z = 0, \qquad (3.9)$$

$$w_2 - v_2 - E\frac{\partial^2 u_2}{\partial z^2} = -\frac{1}{2}\left(v_1\frac{\partial u_1}{\partial y} + w_1\frac{\partial u_1}{\partial z}\right) - \frac{1}{2}\left(\bar{v}_1\frac{\partial u_1}{\partial y} + \bar{w}_1\frac{\partial u_1}{\partial z}\right), \qquad (3.10)$$

$$u_2 + Ri_c\frac{\partial p_2}{\partial y} - E\frac{\partial^2 v_2}{\partial z^3} = -\frac{1}{2}\left(v_1\frac{\partial v_1}{\partial y} + w_1\frac{\partial v_1}{\partial z}\right) - \frac{1}{2}\left(\bar{v}_1\frac{\partial v_1}{\partial y} + \bar{w}_1\frac{\partial v_1}{\partial z}\right), \qquad (3.11)$$

$$\partial p_2/\partial z = \theta_2, \qquad (3.12)$$

$$w_2 - \frac{v^2}{Ri_c} - \frac{E}{\sigma}\frac{\partial^2\theta_2}{\partial z^2} = -\frac{1}{2}\left(v_1\frac{\partial \theta_1}{\partial y} + w_1\frac{\partial \theta_1}{\partial z}\right) - \frac{1}{2}\left(\bar{v}_1\frac{\partial \theta_1}{\partial y} + \bar{w}_1\frac{\partial \theta_1}{\partial z}\right), \qquad (3.13)$$

where the bars denote complex conjugates. To a first approximation the terms on the right-hand sides of (3.10), (3.11) and (3.13) reduce to

$$E^{\frac{1}{3}}|A|^2\left(\frac{\lambda + l}{2\lambda^2 l}\right)m\pi\sin 2m\pi z, \quad E^{\frac{2}{3}}|A|^2\frac{\lambda}{l}m\pi\sin 2m\pi z,$$
$$E^{\frac{1}{3}}|A|^2\left(\frac{\lambda + Ri_c l}{2Ri_c\lambda^2 l}\right)m\pi\sigma\sin 2m\pi z,$$

respectively. A particular integral of (3.9)–(3.13) is then

$$\left.\begin{aligned} w_2 \equiv 0, \quad v_2 &= -E^{\frac{1}{3}}|A|^2[(\lambda + l)/2\lambda^2 l]\,m\pi\sin 2m\pi z,\\ u_2 &= E^{\frac{2}{3}}|A|^2\frac{\lambda}{l}m\pi\sin 2m\pi z,\\ \theta_2 &= -E^{-\frac{2}{3}}|A|^2\sigma\frac{[\sigma(\lambda + Ri_c l) - (\lambda + l)]}{8m\pi Ri_c\lambda^2 l}\sin 2m\pi z. \end{aligned}\right\} \qquad (3.14)$$

The $O(|\Delta|^{\frac{3}{2}})$ problem is

$$\partial v_3/\partial y + \partial w_3/\partial z = 0, \qquad (3.15)$$

$$w_3 - v_3 - E\frac{\partial^2 u_3}{\partial z^2} = -\frac{\partial u_1}{\partial T} - \left(v_2\frac{\partial u_1}{\partial y} + w_1\frac{du_2}{dz}\right), \qquad (3.16)$$

$$u_3 + Ri_c\frac{\partial p_3}{\partial y} - E\frac{\partial^2 v_3}{\partial z^2} = -\frac{\partial v_1}{\partial T} - \left(v_2\frac{\partial v_1}{\partial y} + w_1\frac{dv_2}{dz}\right) + \operatorname{sgn}\Delta\frac{\partial p_1}{\partial y}, \qquad (3.17)$$

$$\partial p_3/\partial z = \theta_3, \tag{3.18}$$

$$w_3 - \frac{v_3}{Ri_c} - \frac{E}{\sigma}\frac{\partial^2\theta_3}{\partial z^2} = -\frac{\partial\theta_1}{\partial T} - \left(v_2\frac{\partial\theta_1}{\partial y} + w_1\frac{d\theta_2}{dz}\right) + \operatorname{sgn}\Delta\frac{v_1}{Ri_c^2}. \tag{3.19}$$

The inhomogeneities on the right-hand sides of these equations may be evaluated in terms of the lower-order solutions. Suppose that there is a solution of the form

$$w_3 = W\mathscr{E}\sin m\pi z, \quad \text{etc.,}$$

where W is a constant, i.e. that the inhomogeneities force a resonance. In order to keep our expansion uniformly valid in time we must find a condition that W be zero. Substituting this form into (3.15)–(3.19) we find that to leading order

$$lV + \lambda W = 0, \tag{3.20}$$

$$W - V + E^{\frac{1}{3}}\lambda^2 U = \dot{A}(\lambda + l)/\lambda^2 l = Q_1, \tag{3.21}$$

$$u + ilE^{-\frac{1}{3}}Ri_c P + E^{\frac{1}{3}}\lambda^2 V = E^{\frac{1}{3}}\frac{\lambda}{l}\dot{A} - \operatorname{sgn}\Delta\frac{\sigma(\lambda + Ri_c l)}{Ri_c\lambda^3}A = Q_2, \tag{3.22}$$

$$i\lambda E^{-\frac{1}{3}}P = \Theta, \tag{3.23}$$

$$W - \frac{V}{Ri_c} + \frac{\lambda^2 E^{\frac{1}{3}}}{\sigma}\Theta = \frac{\sigma(\lambda + lR_c)}{Ri_c\lambda^2 l}\dot{A} - \operatorname{sgn}\Delta\frac{\lambda}{Ri_c^2 l}AE^{\frac{1}{3}}$$
$$- E^{-\frac{1}{3}}\sigma\frac{\sigma(\lambda + R_c l) - (\lambda + l)}{16Ri_c\lambda^2 l}A|A|^2 = Q_3. \tag{3.24}$$

Elimination of U, V, P and Θ leads to an equation for W which has the solution $W \equiv 0$ when

$$\lambda Q_1 - E^{\frac{1}{3}}\lambda^3 Q_2 + \sigma Ri_c l Q_3 = 0,$$

i.e.

$$\dot{A}\left[\frac{l+\lambda}{l\lambda} + \frac{\sigma^2(\lambda + Ri_c l)}{\lambda^2}\right] + \operatorname{sgn}(\Delta)E^{\frac{1}{3}}A\sigma l - \sigma^2 E^{-\frac{1}{3}}|A|^2 A\left[\frac{\sigma(\lambda + Ri_c l) - (l + \lambda)}{16\lambda^2}\right] = 0.$$

Using $\lambda/l = -\frac{1}{2}(1+\sigma)$ and $Ri_c = (1+\sigma)^2/4\sigma$ we obtain

$$\dot{A} = \operatorname{sgn}\Delta\frac{4E^{\frac{1}{3}}\sigma\lambda^2}{(1+\sigma)(1-\sigma)^2}A - \frac{E^{-\frac{1}{3}}\sigma^2}{16(1+\sigma)}A|A|^2. \tag{3.25}$$

A more convenient form is obtained by writing $A = E^{\frac{1}{3}}\bar{A}$ and $T = E^{\frac{1}{3}}\tau$. Then (3.25) yields

$$d|\bar{A}|^2/d\tau = \operatorname{sgn}(\Delta)\,\alpha|A|^2 - \beta|A|^4, \tag{3.26}$$

where
$$\alpha = \frac{8\sigma}{(1+\sigma)(1-\sigma)^2}\lambda^2, \quad \beta = \frac{\sigma^2}{8(1+\sigma)}.$$

This is identical to the form of equation given by Stuart (1960), whose solution is

$$|\bar{A}|^2 = \frac{|\bar{A}(0)|^2\exp(\alpha\operatorname{sgn}(\Delta)\tau)}{1 + \beta/\alpha|\bar{A}(0)|^2[\exp(\alpha\operatorname{sgn}(\Delta)\tau) - 1]},$$

where $\bar{A}(0)$ is some initial value of $\bar{A}$. When $\operatorname{sgn}\Delta < 0$, that is the perturbation of the Richardson number is above the critical value, the flow is stable in the linear analysis and $|\bar{A}|^2 \to 0$ as $\tau \to \infty$. In the linearly unstable case, $\operatorname{sgn}\Delta > 0$, it can be seen that

$$|\bar{A}|^2 \to \alpha/\beta = 64\lambda^2/\sigma(1-\sigma)^2 \quad \text{as} \quad \tau \to \infty,$$

i.e., the amplitude tends to a constant value.

4. Discussion

Let F^* be the total meridional flux of zonal momentum due to advection. Then for a Boussinesq fluid

$$F^* = \rho^* H U^2 \int_0^1 uv\, dz = \rho^* H U^2 F.$$

Using the expansion (3.1) and employing the basic solution (3.7) we obtain

$$F = \int_0^1 \{|\Delta| (\tfrac{1}{2} u_1 \bar{v}_1 + u_0 v_2) + O(|\Delta|^{\frac{3}{2}})\}\, dz, \tag{4.1}$$

where averaging over one wavelength in the meridional (y) direction has eliminated terms in $|\Delta|^{\frac{1}{2}}$ and $|\Delta|\, u_1 v_1$.

From (3.7), (3.8) and (3.14) and setting $m = 1$ we see that

$$\begin{aligned} u_1 \bar{v}_1 &= E^{\frac{1}{3}} |A|^2 [(\lambda + l)/\lambda l^2] \sin^2 \pi z \\ &= -E^{\frac{1}{3}} |A|^2 [(1-\sigma)/(1+\sigma)]\, l^{-2} \sin^2 \pi z, \\ u_0 v_2 &= -E^{\frac{1}{3}} \pi z |A|^2 [(1-\sigma)/(1+\sigma)^2]\, l^{-2} \sin 2\pi z, \end{aligned}$$

and (4.1) becomes

$$F = \frac{|\Delta|\, E^{\frac{1}{3}} |A|^2}{4l^2} \left(\frac{1-\sigma}{1+\sigma}\right)^2,$$

which means that zonal momentum is convected polewards for all values of the Prandtl number. Before discussing the significance of this result to the dynamics of Jupiter's atmosphere, we should examine the validity of our approximations.

In §2 we assumed that $\epsilon \ll Ri$ and later on we included terms $O(E^{\frac{2}{3}})$ in the expansion of (2.24) whilst rejecting a correction $O(\epsilon^2)$. Our theory is particularly appropriate therefore for shallow atmospheres. The mid-latitude zonal currents on Jupiter have velocity of order $10\,\mathrm{m\,s^{-1}}$ and $f \sim 10^{-4}\,\mathrm{s^{-1}}$, so that the horizontal length scale U/f is about 10^5 m. Together with the assumption that $Ri \sim 1$, this means that we require $H \ll 10^5$ m and also $H \ll E^{\frac{1}{3}} \times 10^5$ m. Taking a value for the eddy viscosity $\nu \sim 10^2\,\mathrm{m^2\,s^{-1}}$ (Hide 1969), we obtain $E \sim 10^{-3}$. We require therefore $H \ll 10^4$ m, which is rather smaller than the scale height 2×10^4 m, but the extension of our theory to include deeper-atmosphere effects is not expected to be a significant modification. More serious defects in the model are the assumptions that the zonal flow is meridionally unbounded and that $Ri > 0$. There is evidence that the mid-latitude currents are sharply bounded and that Ri may vary with latitude owing to internal heat sources; it may even be negative in some latitudes (Ingersoll & Cuzzi 1969). Indeed, numerical investigation of a convectively unstable atmosphere ($Ri < 0$) by Williams & Robinson (1973) has successfully reproduced some of the features of the visual appearance of the

Jovian atmosphere and has demonstrated that a purely symmetric motion can lead to an equatorial acceleration.

Summarizing, we may say that if our model is an acceptable approximation to the conditions on Jupiter then, at least when the Richardson number is not far removed from the critical value, symmetric instabilities do not convect zonal momentum equatorwards and will not therefore be able to support an equatorial jet. For a more complete description the stability analysis should be carried beyond that of § 3 to more moderate values of Δ† and it is hoped that this will be the subject of a subsequent paper.

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† It is noted that the variation of f with latitude may also be important (Hide 1966; Stone 1971).